Problem 34

145 is a curious number, as 1! + 4! + 5! = 1 + 24 + 120 = 145.

Find the sum of all numbers which are equal to the sum of the factorial of their digits.

Note: as 1! = 1 and 2! = 2 are not sums they are not included.

Solution

Note that $9! == 362\,880$, so an eight-digit number cannot be the sum of its factorials, since $9! \times 8 < 10^7$. If a number doesn't contain the digit 9, then the most the sum can come to is six digits (in fact, can be at most 241920, and hence must be of five digits only, because the first digit can't be an 8).
And a number can't contain a 9 unless it's of six or more digits. Hence we can rule out the numbers from 88889 to 99999.
We can also rule out one-digit numbers, as their factorials are all too big; if a number has two digits, then it can't contain anything above a 4; if it contains a 1 or a 0, then it can't be valid, because the only two-digit factorial is 24, and that isn't one more than something with a 1 or 0 in. Hence it must consist of the digits 2,3,4, with at least one 4 (since it must contain a 3 or a 4, but if it contains no 4s then it contains one 3, and hence it contains another 3 to get it up to two digits, but that's 33 which we're not allowed). So the available numbers are 24,34,44,42,43; it can easily be seen that these are all too big or too small.
Hence the number must be of three or more digits.

If the number contains 9s and wants to be seven digits, then the first digit must be a 2 or less (since $9! \times 7 == 2\,540\,160$) and hence we can't have seven digits. If the number contains 9s and wants to be six digits, then only at most two of the digits can be 9s (since $3 \times 9! > 1\,000\,000$).
Hence we only need to go up to $2 \times 9! + 4 \times 8! == 887\,040$, and we only need to start at $9! + 5 \times 1! == 362\,885$.

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In[94]:= validQ[n_] := Total[IntegerDigits[n]!] == n

In[125]:= Select[Join[Range[100, 88 888], Range[362 885, 887 040]], validQ] // Total

Out[125]= 40 730
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This took nine seconds, but ignored the “at most two 9s and at least one 9” rule.