Problem 9

There exists exactly one Pythagorean triplet for which a + b + c = 1000.
Find the product abc.

Solution

Using built-in functions:

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In[112]:= a b c /.
  Solve[a^2 + b^2 == c^2 && a + b + c == 1000 && Thread[{a, b, c} ∈ Integers] && 0 < a < b < c]

Out[112]= {31 875 000}
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A way that may end up being worse:
$(a+b)^2 - 2\,a\,b == c^2$, so $(1000-c)^2 - 2\,a\,b == c^2$ and $a\,b == 1000\,(500-c)$, from which
$a\,b\,c == 1000\,c\,(500-c)$.
So if we can find the hypotenuse c, then we're done.
We also have $333 < c < 500$ (the top inequality coming from the expression for $a\,b\,c$, the bottom one coming from the fact that c is smallest when $a == b == c$). So there are 166 possible answers.
For each of these answers, we can find the roots of the corresponding cubic equation whose roots are a,b,c. We know that $a\,b\,c == 1000\,c\,(500-c)$, and that $a+b+c == 1000$, and can easily show that $a\,b + b\,c + a\,c == \left(1000 \times 500 - c^2\right)$ from squaring $(a+b+c) == 1000$.

In[119]:= $$\frac{x^3 - 1000\,x^2 + (1000 \times 500 - c\,\hat{}\,2)\,x - 1000\,c\,(500 - c)}{(x - c)} \;//\; \text{Simplify}$$

Out[119]= $500\,000 + c\,(-1000 + x) - 1000\,x + x^2$

Quadratic formula can be used to find the roots; to save tedium, I'll just use Reduce.

In[120]:= $\text{Reduce}\left[500\,000 + c\,(-1000 + x) - 1000\,x + x^2 == 0,\ x\right]$

Out[120]= $x == \frac{1}{2}\left(1000 - c - \sqrt{-1\,000\,000 + 2000\,c + c^2}\right)\ ||\ x == \frac{1}{2}\left(1000 - c + \sqrt{-1\,000\,000 + 2000\,c + c^2}\right)$

So we just need c such that $\sqrt{-1\,000\,000 + 2000\,c + c^2}$ is integer.

In[121]:= $\text{Select}\left[\text{Range}[334, 499],\ \text{Function}\left[\{c\},\ \text{IntegerQ}\left[\sqrt{-1\,000\,000 + 2000\,c + c^2}\right]\right]\right]$

Out[121]= {425}

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In[123]:= 1000 (500 - c) c /. c → 425

Out[123]= 31 875 000
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